HW #3

6.1 #10, 16, 18, 23ceg, 26

6.2 # 2, 4, 5, 8, 20, 25cf

6.1) (10) $2x + y + 3z = 0$ in $\mathbb{R}^3$

Since $(0,0,0)$ is on the plane we can get the 2 basis vectors by finding 2 points a, b such that $a, b, 0$ are not colinear.

2 points are: $v_1 = (1,-2,0)$ and
$v_2 = (0,3,-1)$

So $A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 3 & -1 \end{pmatrix}$. Then

$$null(A) = sp\left\{\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}\right\} = W^{\perp}$$

(16) Proj $(1,2,1)$ on $x+y+z=0$ in $\mathbb{R}^3$

A basis for the plane is:
$v_1 = (1,-1,0)$ and $v_2 = (1,0,-1)$

The ortho. comp. is: $sp\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 0 & -1 & 1 & 1 \end{array}\right) \overset{R_2+R_1}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & 1 & 1 \end{array}\right) \underset{R_3+R_2}{\overset{R_1-R_2}{\sim}} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{array}\right)$$

$$\overset{R_3/3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 4/3 \end{array}\right) \underset{R_2-2R_3}{\overset{R_1+R_3}{\sim}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 4/3 \end{array}\right)$$

So

$$(1,2,1) = \underbrace{\frac{-2}{3}v_1 + \frac{1}{3}v_2}_{W} + \underbrace{\frac{4}{3}v_3}_{W^\perp}$$

Thus

$$b_W = \frac{-2}{3}v_1 + \frac{1}{3}v_2 = (1/3, 2/3, -1/3)$$

(18) Proj $(-1,0,1)$ on $x+y=0$ in $\mathbb{R}^3$

A basis for the plane is:

$v_1 = (1,-1,0)$ & $v_2 = (0,0,1)$

$$W^\perp = sp\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$$

Thus

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right) \overset{R_2 \leftrightarrow R_3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{array}\right) \overset{R_3+R_1}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -1 \end{array}\right)$$

$$\overset{R_3/2}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/2 \end{array}\right) \overset{R_1-R_3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1/2 \end{array}\right)$$

So

$$(-1,0,1) = \underbrace{\frac{-1}{2}v_1 + v_2}_{W} \underbrace{- \frac{1}{2}v_3}_{W^\perp}$$

So $b_W = \frac{-1}{2}v_1 + v_2 = \left(\frac{-1}{2}, \frac{1}{2}, 1\right)$

23 c) True. This is the definition of $W^\perp$.

e) False. If b were orthogonal to every vector of W then if $v_1, \dots, v_k$ is a basis for W:

$$Ab = \begin{pmatrix} - v_1 - \\ \vdots \\ - v_k - \end{pmatrix} b = \begin{pmatrix} v_1 \cdot b \\ \vdots \\ v_k \cdot b \end{pmatrix} = 0$$

So that $b \in \text{null}(A) = W^\perp$
$\Rightarrow b = 0$.

$\times$ since $b \neq 0$ in general.

g) True. Let $v_1, \dots, v_k$ be the basis of W. Then

$$Ab = \begin{pmatrix} - v_1 - \\ \vdots \\ - v_k - \end{pmatrix} b = 0$$

$\Rightarrow b \in \text{null}(A) = W^\perp$
$\Rightarrow b_W = 0$

(26) Let A be an $m \times n$ matrix.

(a)

$$W = \{x \mid xA = 0\} = \{x \mid A^T x^T = 0\}$$

Thus W is the orthogonal complement of the row space of A^T.

$\therefore$ W is a subspace of $\mathbb{R}^m$.

(b) Since the row space of A^T is the column space of A, we have that W is orthogonal to the column space of A.

6.2) 2 $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -1+1=0$ ✓

$$b_w = \frac{(2,1,4)\cdot(-1,0,1)}{(-1,0,1)\cdot(-1,0,1)}(-1,0,1) + \frac{(2,1,4)\cdot(1,1,1)}{(1,1,1)\cdot(1,1,1)}(1,1,1)$$

$$= \frac{-2+0+4}{1+0+1}(-1,0,1) + \frac{2+1+4}{1+1+1}(1,1,1) = (-1,0,1) + \frac{7}{3}(1,1,1)$$

$$= (4/3, 7/3, 10/3)$$

4 $\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = -1-1+1+1=0$ ✓

$\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = 1-1-1+1=0$ ✓

$\begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = -1+1-1+1=0$ ✓

$$b_w = \frac{(1,-1,-1,1)\cdot(2,1,3,1)}{(1,-1,-1,1)\cdot(1,-1,-1,1)}(1,-1,-1,1) + \frac{(-1,1,1,1)\cdot(2,1,3,1)}{(-1,1,1,1)\cdot(-1,1,1,1)}(-1,1,1,1)$$

$$+ \frac{(1,1,-1,1)\cdot(2,1,3,1)}{(1,1,-1,1)\cdot(1,1,-1,1)}(1,1,-1,1) = \frac{1}{4}(1,-1,-1,1) + \frac{3}{4}(-1,1,1,1)$$

$$+ \frac{1}{4}(1,1,-1,1) = \left(\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{5}{4}\right)$$

⑤ The plane $2x+3y+z=0$ has a basis:
$v_1=(1,0,-2)$ & $v_2=(0,1,-3)$.

Let's first make this an orthogonal basis:
Let $a_1=v_1$. Then

$-\frac{15}{5}+\frac{12}{5}$

$$a_2=v_2-\left(\frac{v_2\cdot a_1}{a_1\cdot a_1}a_1\right)=(0,1,-3)-\frac{(0,1,-3)\cdot(1,0,-2)}{(1,0,-2)(1,0,-2)}(1,0,-2)$$

$$=(0,1,-3)-\frac{6}{5}(1,0,-2)=\left(\frac{-6}{5},1,\frac{-3}{5}\right)$$

2
36
25
9
70

Now normalize:

$$w_1=\frac{a_1}{\|a_1\|}=\frac{(1,0,-2)}{\sqrt{1+0+4}}=\left(\frac{1}{\sqrt5},0,\frac{-2}{\sqrt5}\right)$$

$$w_2=\frac{a_2}{\|a_2\|}=\frac{(-6/5,1,-3/5)}{\sqrt{\frac{36}{25}+\frac{25}{25}+\frac{9}{25}}}=\frac{(-6/5,1,-3/5)}{\sqrt{70}/5}=\left(\frac{-6}{\sqrt{70}},\frac{5}{\sqrt{70}},\frac{-3}{\sqrt{70}}\right)$$

Then $\{w_1,w_2\}$ is an orthonormal basis for the plane.

⑧ $W=sp\{(1,1,0),(-1,2,1)\}$

$\frac{\sqrt{22}}{2}$

Let $v_1=(1,1,0)$ and use G-S to get

$$v_2=(-1,2,1)-\frac{(-1,2,1)\cdot(1,1,0)}{(1,1,0)\cdot(1,1,0)}(1,1,0)=(-1,2,1)-\frac{1}{2}(1,1,0)$$

$$=(-3/2,3/2,1)$$

Now normalize:

$$w_1=\frac{v_1}{\|v_1\|}=\frac{(1,1,0)}{\sqrt2}=(1/\sqrt2,1/\sqrt2,0),\quad w_2=\frac{v_2}{\|v_2\|}=\frac{(-3/2,3/2,1)}{\sqrt{\frac94+\frac94+\frac44}}=\left(\frac{-3}{\sqrt{22}},\frac{3}{\sqrt{22}},\frac{2}{\sqrt{22}}\right)$$

Then $\{w_1,w_2\}$ is an orthonormal basis.

(20) $w_1 = \frac{1}{\sqrt{3}}(1,1,1)$

Let $a_1 = (1,1,1)$, $a_2 = (1,-1,0)$, $a_3 = (1,0,-1)$.

Using G-S with $v_1 = a_1$:

$$v_2 = (1,-1,0) - \frac{(1,1,1)\cdot(1,-1,0)}{(1,1,1)\cdot(1,1,1)}(1,1,1) = (1,-1,0) - 0(1,1,1)$$

$$= (1,-1,0)$$

& $$v_3 = (1,0,-1) - \left(\frac{(1,1,1)\cdot(1,0,-1)}{(1,1,1)\cdot(1,1,1)}(1,1,1) + \frac{(1,-1,0)\cdot(1,0,-1)}{(1,-1,0)\cdot(1,-1,0)}(1,-1,0)\right)$$

$$= (1,0,-1) - (0(1,1,1) + \tfrac{1}{2}(1,-1,0) = (\tfrac{1}{2}, \tfrac{1}{2}, -1)$$

Now normalize:

$$w_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}(1,1,1), \quad w_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}(1,-1,0),$$

$$w_3 = \frac{v_3}{\|v_3\|} = \frac{(\frac{1}{2}, \frac{1}{2}, -1)}{\sqrt{\frac{1}{4}+\frac{1}{4}+\frac{4}{4}}} = \frac{1}{\sqrt{6}}(1,1,-2)$$

(25) (c) True. Every subspace has a basis, so use the normalized G-S on it.

(f) True. Consider the standard basis for $\mathbb{R}^n$. Replace one of the basis elements with our unit vector in a way that the basis remains linearly independent. Then use norm. G-S on this set.